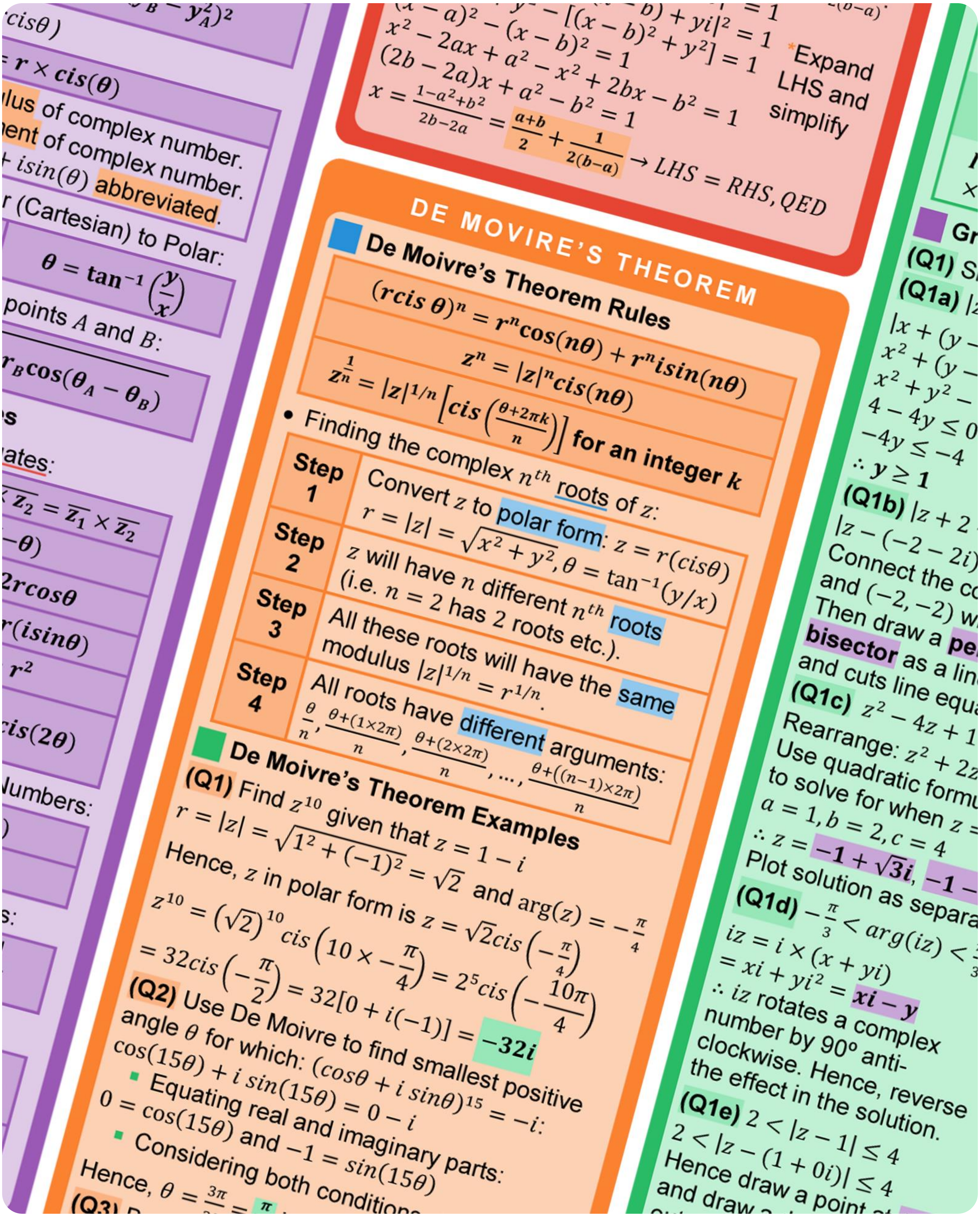

ATAR Mathematics Specialist Units 3 & 4

Exam Notes for Western Australian Year 12 Students

Created by Anthony Bochrinis

Version 3.0 (Updated 21/12/19)

REVISE ONLINE

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▶ About the Author - Anthony Bochrinis

Hello! My name is Anthony and I graduated from high school in 2012, completed a Bachelor of Actuarial Science in 2015, completed my Graduate Diploma in Secondary Education in 2017 and am now a secondary mathematics teacher!

My original exam notes (created in 2013) were inspired by Severus Snape's copy of Advanced Potion Making in Harry Potter and the Half-Blood Prince; a textbook filled with annotations containing all of the pro tips and secrets to help gain a clearer understanding.

Thank you for being a part of my journey in realising that teaching is my lifelong vocation!

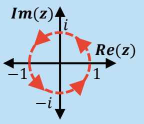
COMPLEX NUMBERS

IMAGINARY NUMBERS

Powers of Imaginary Numbers (i)

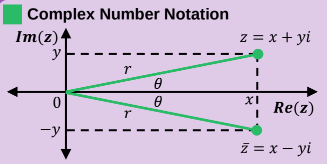
$i^{-4} = 1$	$i^0 = 1$	$i^4 = 1$
$i^{-3} = \sqrt{-1}$	$i^1 = \sqrt{-1}$	$i^5 = \sqrt{-1}$
$i^{-2} = -1$	$i^2 = -1$	$i^6 = -1$
$i^{-1} = -i$	$i^3 = -i$	$i^7 = -i$

- To find value of i^n, divide power by 4:
  - Remainder 0 = 1
  - Remainder 1 = i
  - Remainder 2 = -1
  - Remainder 3 = $-i$



COMPLEX NUMBERS

Complex Number Notation



- Im: imaginary axis (vertical axis → y-axis).
- Re: real axis (horizontal axis → x-axis).
- z: complex number ($z = x + yi$).
- $\bar{z}$: conjugate of a complex number ($\bar{z} = x - yi$) and is reflected in the real axis.
- x: real components (horizontal axis).
- y: imaginary component (vertical axis).
- r: modulus (length) of a complex number and can also be represented by $|z|$.
- θ: argument (angle that the complex number makes with the real axis) of complex number and can also be represented by $\arg(z)$.

Rectangular (Cartesian) Form ($x + yi$)

- Convert Polar to Rectangular (Cartesian):

$x = r \times \cos(\theta)$	$y = r \times \sin(\theta)$

- Distance between two points A and B:

$$\vec{AB} = \sqrt{(x_B^2 - x_A^2)^2 + (y_B^2 - y_A^2)^2}$$

Polar Form ($rcis\theta$)

$$z = r \times cis(\theta)$$

- r : is the modulus of complex number.
- θ : is the argument of complex number.
- $cis(\theta)$: $\cos(\theta) + i\sin(\theta)$ abbreviated.

- Convert Rectangular (Cartesian) to Polar:

$r = \lvert z\rvert = \sqrt{x^2 + y^2}$	$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

- Distance between two points A and B:

$$\vec{AB} = \sqrt{r_A^2 + r_B^2 - 2r_A r_B \cos(\theta_A - \theta_B)}$$

Complex Number Rules

- Rules for Complex Conjugates:

$\overline{z_1 \pm z_2} = \bar{z_1} \pm \bar{z_2}$	$\overline{z_1 \times z_2} = \bar{z_1} \times \bar{z_2}$

$$\bar{z} = x - yi = rcis(-\theta)$$
$$z + \bar{z} = 2Re(z) = 2x = 2rcos\theta$$
$$z - \bar{z} = 2iIm(z) = 2yi = 2r(isin\theta)$$
$$z \times \bar{z} = x^2 + y^2 = |z|^2 = r^2$$
$$\frac{z}{\bar{z}} = \left(\frac{x^2 - y^2}{x^2 + y^2}\right) + i\left(\frac{2xy}{x^2 + y^2}\right) = cis(2\theta)$$

- Rules for Arguments of Complex Numbers:

$$\arg(z \times w) = \arg(z) + \arg(w)$$
$$\arg(z \div w) = \arg(z) - \arg(w)$$

- Rules for Moduli of Complex Numbers:

$\lvert z \times w\rvert = \lvert z\rvert \times \lvert w\rvert$	$\left\lvert\frac{z}{w}\right\rvert = \frac{\lvert z\rvert}{\lvert w\rvert}$

- Simplifying Complex Numbers:

$$z^{-1} = \frac{1}{z} = \frac{1}{x + yi} \times \frac{x - yi}{x - yi} = \frac{x - yi}{x^2 + y^2} = \frac{\bar{z}}{|z|}$$
$$\frac{z}{w} = \frac{a + bi}{c + di} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di} = \frac{z \times \bar{w}}{|w|^2}$$

COMPLEX NUMBER ALGEBRA

Complex Number Algebra Examples

(Q1) Express $\frac{4+3i}{2-i}$ in cartesian form:

$$\frac{4 + 3i}{2 - i} = \frac{4 + 3i}{2 - i} \times \frac{2 + i}{2 + i} = \frac{(4 + 3i) \times (2 + i)}{(2 - i) \times (2 + i)}$$
$$= \frac{8 + 4i + 6i + 3i^2}{4 - i^2} = \frac{5 + 10i}{5} = \mathbf{1 + 2i}$$

(Q2) Express $(-\sqrt{3} + i)(4 + 4i)$ in polar form:

- Converting $(-\sqrt{3} + i)$ to polar form:

$r = |z| = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = \sqrt{3 + 1} = \sqrt{4} = 2$

$\theta = \arg(z) = tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = -\frac{\pi}{6}$ but as z is in the second quadrant, $\arg(z) = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$

► Topic Is Continued In Next Column ◄

COMPLEX NUMBER ALGEBRA

Complex Number Algebra Examples

(Q2) Express $(-\sqrt{3} + i)(4 + 4i)$ in polar form:

- Converting $(4 + 4i)$ to polar form:

$r = |z| = \sqrt{4^2 + 4^2} = \sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$

$\theta = \arg(z) = tan^{-1}\left(\frac{4}{4}\right) = \frac{\pi}{4}$, z is in first quadrant.

- Multiplying two complex numbers together:

$\left[2cis\left(\frac{5\pi}{6}\right)\right] \times \left[4\sqrt{2}cis\left(\frac{\pi}{4}\right)\right] = 8\sqrt{2}cis\left(\frac{5\pi}{6} + \frac{\pi}{4}\right)$

$= 8\sqrt{2}cis\left(\frac{26\pi}{24}\right) = \mathbf{8\sqrt{2}cis\left(\frac{13\pi}{12}\right)}$

(Q3) Determine all roots, real and complex, of the equation $f(z) = z^3 - 4z^2 + z + 26$:

Substitute different values of z until $f(z) = 0$:
$f(0) = 26 \neq 0$, $f(1) = 24 \neq 0$, $f(-1) = 20 \neq 0$, $f(2) = 20 \neq 0 \rightarrow$ these are not factors
$f(-2) = 0$ hence $(z + 2)$ is a factor
$\therefore z^3 - 4z^2 + z + 26 = (z + 2)(z^2 + bz + c)$
Using polynomial long division (on page 2):

$propFrac\left(\frac{z^3 - 4z^2 + z + 26}{z + 2}\right) = z^2 - 6z + 13$

Find roots of $z^2 - 6z + 13$ by quadratic formula:

$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2(1)}$

$= \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm \sqrt{16}\sqrt{-1}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$

Hence roots are $z = \mathbf{-2, 3 + 2i, 3 - 2i}$

(Q4) Find all the complex numbers that satisfy the equation $|z|^2 - iz = 36 + 4i$:

- Let $z = x + yi$ and hence:

$|(x + yi)|^2 - i(x + yi) = 36 + 4i$
$\left(\sqrt{x^2 + y^2}\right)^2 - xi - yi^2 = 36 + 4i$
$x^2 + y^2 - xi + y - 36 - 4i = 0$

*Expand and simplify LHS and RHS

- Equating real and imaginary parts:

$x^2 + y^2 + y - 36 = 0$ and $-x - 4 = 0$
Hence, $x = -4$ and $(-4)^2 + y^2 + y - 36 = 0$
$16 + y^2 + y - 36 = 0$
$y^2 + y - 20 = 0$ and $(y + 5)(y - 4) = 0$
Giving $y = -5, 4$ hence $z = \mathbf{-4 - 5i, -4 + 4i}$

(Q5) a & b are real & $a \neq b$. If $z = x + yi$ and $|z - a|^2 - |z - b|^2 = 1$, prove $x = \frac{a+b}{2} + \frac{1}{2(b-a)}$:

$|(x + yi) - a|^2 - |(x + yi) - b|^2 = 1$
$|(x - a) + yi|^2 - |(x - b) + yi|^2 = 1$
$(x - a)^2 + y^2 - [(x - b)^2 + y^2] = 1$
$(x - a)^2 - (x - b)^2 = 1$
$x^2 - 2ax + a^2 - x^2 + 2bx - b^2 = 1$
$(2b - 2a)x + a^2 - b^2 = 1$
$x = \frac{1 - a^2 + b^2}{2b - 2a} = \frac{a + b}{2} + \frac{1}{2(b - a)} \rightarrow LHS = RHS, QED$

*Expand LHS and simplify

DE MOIVRE'S THEOREM

De Moivre's Theorem Rules

$$(rcis\ \theta)^n = r^n\cos(n\theta) + r^n i\sin(n\theta)$$
$$z^n = |z|^n cis(n\theta)$$
$$z^{\frac{1}{n}} = |z|^{1/n}\left[cis\left(\frac{\theta + 2\pi k}{n}\right)\right] \text{ for an integer } k$$

- Finding the complex n^{th} roots of z:

Step 1	Convert z to polar form: $z = r(cis\theta)$ $r = \lvert z\rvert = \sqrt{x^2 + y^2}, \theta = \tan^{-1}(y/x)$
Step 2	z will have n different n^{th} roots (i.e. $n = 2$ has 2 roots etc.).
Step 3	All these roots will have the same modulus $\lvert z\rvert^{1/n} = r^{1/n}$.
Step 4	All roots have different arguments: $\frac{\theta}{n}, \frac{\theta + (1 \times 2\pi)}{n}, \frac{\theta + (2 \times 2\pi)}{n}, \ldots, \frac{\theta + ((n-1) \times 2\pi)}{n}$

De Moivre's Theorem Examples

(Q1) Find z^{10} given that $z = 1 - i$

$r = |z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ and $\arg(z) = -\frac{\pi}{4}$

Hence, z in polar form is $z = \sqrt{2}cis\left(-\frac{\pi}{4}\right)$

$z^{10} = \left(\sqrt{2}\right)^{10} cis\left(10 \times -\frac{\pi}{4}\right) = 2^5 cis\left(-\frac{10\pi}{4}\right)$

$= 32cis\left(-\frac{\pi}{2}\right) = 32[0 + i(-1)] = \mathbf{-32i}$

(Q2) Use De Moivre to find smallest positive angle θ for which: $(cos\theta + i\ sin\theta)^{15} = -i$:

$\cos(15\theta) + i\ sin(15\theta) = 0 - i$

- Equating real and imaginary parts:

$0 = \cos(15\theta)$ and $-1 = sin(15\theta)$

- Considering both conditions, $15\theta = \frac{3\pi}{2}$

Hence, $\theta = \frac{3\pi}{30} = \frac{\pi}{10}$ is smallest positive angle.

(Q3) By expanding $(cos\theta + i\ sin\theta)^3$ and simplifying, show that $cos^3\theta = \frac{1}{4}cos3\theta + \frac{3}{4}cos\theta$

- Expand the bracket of $(cos\theta + i\ sin\theta)^3$:

$= cos^3\theta + 3cos^2\theta(isin\theta) + 3\cos(isin\theta)^2 + (isin\theta)^3$
$= cos^3\theta + 3icos^2\theta sin\theta - 3cos\theta\ sin^2\theta - isin^3\theta$

- Simplify $(cos\theta + i\ sin\theta)^3$ using De Moivre:

$(cos\theta + i\ sin\theta)^3 = cos3\theta + i\ sin3\theta$

- Equating real parts from both equations:

$\cos^3\theta - 3cos\theta\sin^2\theta = cos3\theta$
$\cos^3\theta = cos3\theta + 3cos\theta(1 - \cos^2\theta)$
$\cos^3\theta = cos3\theta + 3cos\theta - 3\cos^3\theta$
$4\cos^3\theta = cos3\theta + 3cos\theta$
$\cos^3\theta = \frac{1}{4}cos3\theta + \frac{3}{4}cos\theta \rightarrow LHS = RHS, QED$

*Rearrange and Solve

(Q4) Simplify $\left(cis\left(\frac{3\pi}{4}\right)\right)^{-4} \times \left(\frac{1 + i}{1 - i}\right)^2 \div \sqrt{cis(2\pi)}$

$\frac{cis(-3\pi) \times \left(-\frac{4}{4}\right)}{(cis(2\pi))^{\frac{1}{2}}} = \frac{-1 \times cis(-3\pi)}{cis(\pi)} = -cis(0)$
$= -cis(-3\pi - \pi) = \mathbf{-1}$

► Topic Is Continued In Next Column ◄

DE MOIVRE'S THEOREM

De Moivre's Theorem Examples

(Q4) Find and graph all the complex fourth roots of -16 on an argand plane.

$r = |-16| = \sqrt{(-16)^2} = 16$ and $\arg(-16) = \pi$
Hence, -16 in polar form is $z = 16cis(\pi)$
We need 4 roots hence $n = 4$ and the roots are:

$z_1 = 16^{\frac{1}{4}}cis\left(\frac{\pi}{4}\right) = \mathbf{2cis\left(\frac{\pi}{4}\right)}$

$z_2 = 16^{\frac{1}{4}}cis\left(\frac{\pi + (1 \times 2\pi)}{4}\right)$

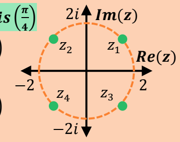
$\therefore z_2 = \mathbf{2cis\left(\frac{3\pi}{4}\right)}$

$z_3 = 16^{\frac{1}{4}}cis\left(\frac{\pi + (2 \times 2\pi)}{4}\right)$

$\therefore z_3 = \mathbf{2cis\left(\frac{5\pi}{4}\right)}$

$z_4 = 16^{\frac{1}{4}}cis\left(\frac{\pi + (3 \times 2\pi)}{4}\right)$

$\therefore z_4 = \mathbf{2cis\left(\frac{7\pi}{4}\right)}$



*All roots are equally spaced out by an angle of $\frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2}$

(Q5) One of the solutions of $z^3 = a$, for some constant a, is $z = 4\sqrt{3} - 4i$. Determine all other solutions in Cartesian form.

$r^{1/3} = |4\sqrt{3} - 4i| = \sqrt{\left(4\sqrt{3}\right)^2 + (-4)^2} = 8$ and

$\arg(4\sqrt{3} - 4i) = \tan^{-1}\left(\frac{4}{-4\sqrt{3}}\right) = -\frac{\pi}{6}$

Hence, $4\sqrt{3} - 4i$ in polar form is $z = 8cis\left(-\frac{\pi}{6}\right)$

$z_1 = 8cis\left(-\frac{\pi}{6}\right) = \mathbf{4\sqrt{3} - 4i}$ * $n = 3$ ∴ 3 roots

$z_2 = 8cis\left(-\frac{\pi}{6} + \frac{2\pi}{3}\right) = 8cis\left(\frac{3\pi}{6}\right) = \mathbf{8i}$

$z_3 = 8cis\left(-\frac{\pi}{6} + \frac{4\pi}{3}\right) = 8cis\left(\frac{7\pi}{6}\right) = \mathbf{-4\sqrt{3} - 4i}$

ARGAND PLANE GRAPHS

Argand Plane Transformations

Variable	Transformation Description
$z \times i$	Rotates a complex number by 90° anti-clockwise.
$z \times i^n$	Rotates a complex number by $(n\pi/2)$ anti-clockwise.
$z \times n$	Increases modulus of complex number by scale factor n.
$Re(z) \times -1$	Reflects a complex number in the y-axis (impacts $Re(z)$ only).
$Im(z) \times -1$	Reflects a complex number in the x-axis (impacts $Im(z)$ only).

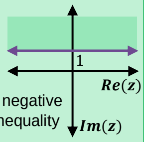
Graphing Complex Numbers

(Q1) Sketch the following in the argand plane:

(Q1a) $|z - 2i| \leq |z|$

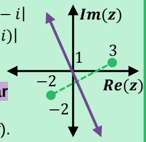
$|x + (y - 2)i| \leq |x^2 + y^2|$
$x^2 + (y - 2)^2 \leq x^2 + y^2$
$x^2 + y^2 - 4y + 4 \leq x^2 + y^2$
$4 - 4y \leq 0$
$-4y \leq -4$
$\therefore y \geq 1$

*Dividing by a negative reverses the inequality



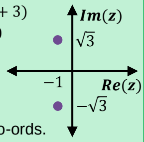
(Q1b) $|z + 2 + 2i| = |z - 3 - i|$

$|z - (-2 - 2i)| = |z - (3 + i)|$
Connect the co-ords $(3,1)$ and $(-2, -2)$ with a line. Then draw a perpendicular bisector as a line (i.e. 90° and cuts line equally in half).



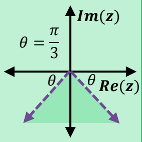
(Q1c) $z^2 - 4z + 1 = -(6z + 3)$

Rearrange: $z^2 + 2z + 4 = 0$
Use quadratic formula to solve for when $z = 0$:
$a = 1, b = 2, c = 4$
$\therefore z = \mathbf{-1 + \sqrt{3}i}, \mathbf{-1 - \sqrt{3}i}$
Plot solution as separate co-ords.



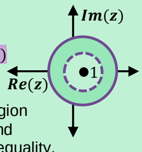
(Q1d) $-\frac{\pi}{3} < arg(iz) < \frac{\pi}{3}$

$iz = i \times (x + yi)$
$= xi + yi^2 = \mathbf{xi - y}$
$\therefore iz$ rotates a complex number by 90° anti-clockwise. Hence, reverse the effect in the solution.



(Q1e) $2 < |z - 1| \leq 4$

$2 < |z - (1 + 0i)| \leq 4$
Hence draw a point at $\mathbf{(1, 0)}$ and draw a doughnut with outer radius of $\mathbf{4}$ and inner radius of $\mathbf{2}$. Shade inner region with inner radius dashed and outer radius solid due to inequality.



FUNCTIONS

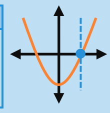
TYPES OF FUNCTIONS

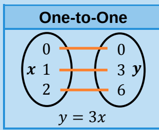
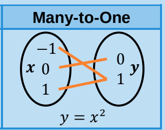
Definition of a Function

- A function satisfies any of the following:

Passes Vertical Line Test

If all possible vertical lines drawn at all points along the curve cut the curve once, it passes the vertical line test.



One-to-One	Many-to-One
	

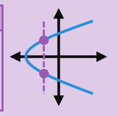
TYPES NON-FUNCTIONS

Definition of a Non-Function

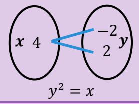
- A non-function (a.k.a. a relation) satisfies:

Fails Vertical Line Test

If all vertical lines drawn at all points along the curve cut the curve more than once, it fails the vertical line test.



Many-to-One



$y^2 = x \rightarrow y = \pm\sqrt{x}$
$y = \sqrt{x}$ and $y = -\sqrt{x}$
i.e. two functions plotted together

COMPOSITE FUNCTIONS

Composite Function Notation ($f \circ g(x)$)

- Applies one function to the results of another.

$$f \circ g(x) = f(g(x))$$

Composite Function Domain and Range

Step 1	Find the natural domain of the inside function of $f \circ g(x)$, $g(x)$.
Step 2	Determine the composite function $f \circ g(x)$ and determine its domain.
Step 3	Domain of $f \circ g(x)$ is intersection of the domains found in steps 1 and 2.
Step 4	Analyse critical points from domain to find the range of $f \circ g(x)$: • Critical points that are $\leq, \geq$ substitute directly into $f \circ g(x)$. • For critical points that are $\neq, <, >$ substitute a number that's slightly higher and lower into $f \circ g(x)$. • Substitute $\infty, -\infty$ into $f \circ g(x)$.

Complex Number Algebra Examples

(Q1) Let $f(x) = ln(x^2 + 1)$ and $g(x) = 2\sqrt{x}$:

(Q1a) Find the composite function $f \circ g(x)$:

$= f\left(2\sqrt{x}\right) = ln\left[\left(2\sqrt{x}\right)^2 + 1\right] = \mathbf{ln(4x + 1)}$

(Q1b) Find $g(x)$ given $f \circ g(x)$ and $f(x)$:

$f(g(x)) \rightarrow \ln(4x + 1) = ln(g(x)^2 + 1)$
Hence $g(x)^2 = 4x$ and $g(x) = \sqrt{4x} = \mathbf{2\sqrt{x}}$

(Q1c) Find $f(x)$ given $f \circ g(x)$ and $g(x)$

$g(x) = 2\sqrt{x} = u$, solve $2\sqrt{x} = u$ for x: $x = \left(\frac{u}{2}\right)^2$
$f(g(x)) = \ln(4x + 1) = \ln[4(u/2)^2 + 1]$
$= \ln(u^2 + 1) \therefore f(u) = \ln(u^2 + 1)$
Change u to x: $f(x) = \mathbf{ln(x^2 + 1)}$

(Q2) Let $f(x) = 1 + \sqrt{x - 2}$ and $g(x) = \frac{1}{x - 5}$, find the domain and range of $g \circ f(x)$.

$g \circ f(x) = g(f(x)) = \frac{1}{f(x) - 5} = \frac{1}{1 + \sqrt{x - 2} - 5} = \frac{1}{\sqrt{x - 2} - 4}$

- Finding domain of inside function $f(x)$:

Domain of $f(x) = \{x \in \mathbb{R}: x \geq 2\}$

- Finding domain of $g \circ f(x)$:

Solve $\sqrt{x - 2} - 4 \neq 0$, $x - 2 \neq 16$, $x \neq 18$
Natural domain of $g \circ f(x) = \mathbf{\{x \in \mathbb{R}: x \neq 18\}}$

- Finding intersection of both domains:

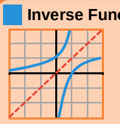
Domain of $g \circ f(x) = \{x \in \mathbb{R}: x \geq 2, x \neq 18\}$

- Analysing critical points from the domain:

Test at point $x = 2$ as $x \neq 18$ $g \circ f(2) = -0.25$
$g \circ f(17.999) \rightarrow -\infty$ and $g \circ f(18.001) \rightarrow \infty$
$g \circ f(-\infty) = undefined$ and $g \circ f(\infty) \rightarrow 0$
Range of $g \circ f(x) = \mathbf{\{y \in \mathbb{R}: y \leq -0.25, y > 0\}}$

INVERSE FUNCTIONS

Inverse Functions ($f^{-1}(x)$)



- When inverse functions are plotted together, they are symmetrical about a 45° line (i.e. the function $y = x$).

$$\textbf{Domain } f(x) = \textbf{Range } f^{-1}(x)$$
$$\textbf{Range } f(x) = \textbf{Domain } f^{-1}(x)$$
$$f \circ f^{-1}(x) = f(f^{-1}(x)) = x$$

Determining the Inverse of a Function

Step 1	Rearrange the function to make x the subject instead of y.
Step 2	Swap the variables x and y, this is the inverse function, $f^{-1}(x)$.

Inverse Function Examples

(Q1) Determine $f^{-1}(x)$ of $f(x) = ln(x + 3) + 1$

$f(x) = y = \ln(x + 3) + 1 \rightarrow y - 1 = \ln(x + 3)$
$e^{y-1} = x + 3 \rightarrow e^{y-1} - 3 = x \rightarrow \mathbf{y = e^{x-1} - 3}$

(Q2) Prove that $f(x) = 2x - 3$ and $g(x) = 0.5x + 1.5$ are inverse functions.

$f(g(x)) = 2(0.5x + 1.5) - 3 = x + 3 - 3 = \mathbf{x}$

ATAR Math Specialist
Units 3 & 4 Exam Notes

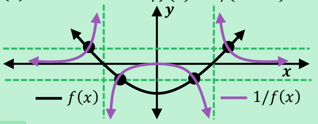
RECIPROCAL FUNCTIONS

Sketching Reciprocal Functions

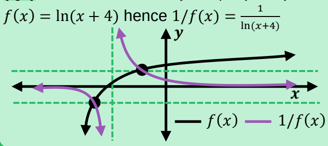
- Sketching the graph $1/f(x)$ given $f(x)$:
  - Any x-intercepts on the graph of $f(x)$ are vertical asymptotes on $1/f(x)$.
  - Any intersections that $f(x)$ has with $y = 1$ or $y = -1$ are points on $1/f(x)$.
  - As $f(x)$ approaches ∞ or $-\infty$ it moves toward the x-axis on $1/f(x)$.

Reciprocal Functions Examples

(Q1) Sketch the function $y = 1/(x^2 - 2)$
$f(x) = x^2 - 2$ hence $1/f(x) = 1/(x^2-2)$



(Q2) Sketch the function $y = 1/\ln(x+4)$
$f(x) = \ln(x+4)$ hence $1/f(x) = \frac{1}{\ln(x+4)}$



ABSOLUTE VALUE FUNCTIONS

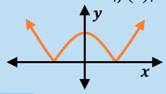
Absolute Value Functions and Notation

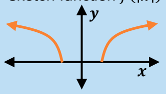
$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$\lvert f(x)\rvert$	Any points below the x-axis are reflected in x-axis and any points above the x-axis aren't changed.
$f(\lvert x\rvert)$	Reflects functions that can't have negative x values (e.g. square root functions) in the y-axis.

Absolute Value Function Examples

(Q1) If $f(x) = x^2 - 3$, sketch function $|f(x)|$
(Q2) If $f(x) = \sqrt{x-2}$, sketch function $f(|x|)$

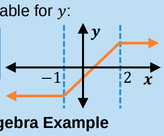




(Q3) Sketch $y = |x+1| - |x-2|$:
Solve each individual absolute value brackets for when it equals each individual absolute value brackets for when it equals 0:
$|x+1| = 0,\ x = -1$ and $|x-2| = 0,\ x = 2$
Hence, $x = -1, 2$ are the critical values.
Create a x/y table with each critical value above. Insert columns between each critical value and choose a random number between them. Solve the entire table for y:

x	-2	-1	0	2	3
y	-3	-3	-1	3	3

*−1 & 2 critical values



Absolute Value Algebra Example

(Q1) If $f(x) = x + 2$ and $g(x) = (x+1)^2 - 5$, solve the equation $|f(x)| = |g(x)|$:
$|g(x)| = |x^2 + 2x - 4| = |x + 2| = |f(x)|$

- Solve for when absolute value is positive:

$x^2 + 2x - 4 = x + 2 \to x^2 + x - 6 = 0$
$(x+3)(x-2) = 0 \to x = -3, 2$

- Solve for when absolute value is negative:

$x^2 + 2x - 4 = -x - 2 \to x^2 + 3x - 2 = 0$
$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-3\pm\sqrt{9+8}}{2} = \frac{-3\pm\sqrt{17}}{2} = 0.56, -3.56$
$\therefore$ x is union of answers $x = -3, 2, 0.56, -3.56$

POLYNOMIAL LONG DIVISION

Polynomial Long Division

Step 1	Divide highest order polynomial in the divisor and dividend and write as the first term in the quotient. Then multiply this by the divisor.
Step 2	Subtract two equations from each other, writing answer underneath.
Step 3	Repeat steps 1 and 2 until a single number remains.

ClassPad Main App Long Division

$Action \to Fraction \to propFrac\left(\frac{A}{B}\right)$

Polynomial Long Division Example

(Q1) Determine $\frac{3x^3-5x^2+10x-3}{3x+1}$

$$\begin{array}{r} x^2-2x+4 \\ 3x+1\,)\overline{3x^3-5x^2+10x-3} \\ -\;\underline{3x^3+1x^2} \\ -6x^2+10x \\ -\;\underline{-6x^2-2x} \\ +12x-3 \\ -\;\underline{+12x+4} \\ -7 \end{array}$$

*$3x^3 \div 3x = x^2$ This is first term in quotient.
*$x^2(3x+1)$ Write under and subtract.
*Remainder −7 (fraction with dividend as denominator).

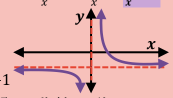
$\frac{3x^3-5x^2+10x-3}{3x+1} = x^2 - 2x + 4 - \frac{7}{3x+1}$

POLYNOMIAL FRACTION FUNCTIONS

Sketching Polynomials Examples

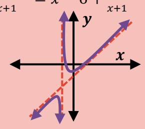
(Q1) Sketch $y = (-3 + 4x - x^2)/(x^2 - x)$
$= \frac{-(x^2-4x+3)}{x(x-1)} = \frac{-(x-3)(x-1)}{x(x-1)} = \frac{-(x-3)}{x} = \frac{3-x}{x} = \frac{3}{x} - 1$

- Function vertical asymptote at $x = 0$
- Function horizontal asymptote at $y = -1$



(Q2) Sketch $y = (x^2 - 5x + 6)/(x+1)$
Long division: $propFrac\frac{x^2-5x+6}{x+1} = x - 6 + \frac{12}{x+1}$

- Function oblique asymptote at $y = x - 6$ (equal to the quotient without the remainder)
- Function vertical asymptote @ $x = -1$



PARTIAL FRACTIONS

Partial Fraction Decomposition

Factor in Denominator	Term in Partial Fraction Decomposition
$ax + b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b} + \cdots + \frac{A_k}{(ax+b)^k}$
$ax^2 + bx + c$	$\frac{Ax+B}{ax^2+bx+c}$

Partial Fraction Examples

(Q1) Simplify $(3x+11)/(x^2-x-6)$
$\frac{3x+11}{x^2-x-6} = \frac{3x+11}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$
$\frac{3x+11}{x^2-x-6} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$ *Equate coefficients and solve
$3x + 11 = A(x+2) + B(x-3)$
$3x + 11 = Ax + 2A + Bx - 3B$
Hence, $3 = A + B$ and $11 = 2A - 3B$
Simultaneously solve: $A = 4, B = -1$

(Q2) Simplify $(x^2 - 29x + 5)/(x-4)^2(x^2+3)$
$\frac{x^2-29x+5}{(x-4)^2(x^2+3)} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{Cx+D}{x^2+3}$
$x^2 - 29x + 5 = A(x-4)(x^2+3) + B(x^2+3) + (Cx+D)(x-4)^2$ *Expand and then simplify
$= (A+C)x^3 + (-4A+B-8C+D)x^2 + (3A+16C-8D)x - 12A + 3B + 16D$

$$\left.\begin{array}{ll} x^3: & A + C = 0 \\ x^2: & -4A + B - 8C + D = 1 \\ x^1: & 3A + 16C - 8D = -29 \\ x^0: & -12A + 3B + 16D = 5 \end{array}\right\} \Rightarrow \begin{array}{l} A = 1 \\ B = -5 \\ C = -1 \\ D = 2 \end{array}$$

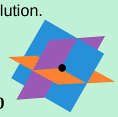
3-D VECTORS

SYSTEMS OF LINEAR EQUATIONS

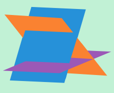
Solutions of Linear Equations

- Echelon matrix form: each leading entry (*i.e.* the first non-zero element in each row) is a column to the right of the previous row.
- Infinite solutions: more than one solution.
  - Graphical representation: the 3 planes produce an intersection that is a line.
  - Echelon matrix last row: $[0\ \ 0\ \ 0\ \ |\ \ 0]$

- Unique solution: only one solution.
  - Graphical representation: the 3 planes produce an intersection that is a line.
  - Echelon matrix last row: $[0\ \ 0\ \ A\ \ |\ \ B]\ A, B \neq 0$



- No solution: zero solutions.
  - Graphical representation: no planes have a common point of intersection.
  - Echelon matrix last row: $[0\ \ 0\ \ 0\ \ |\ \ B]\ B \neq 0$



ClassPad Main App Echelon Form

$Action \to Matrix \to ref([matrix])$

Systems of Linear Equations Examples

(Q1) Reduce this matrix to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 7+a & 5 \\ 2 & 3 & a^2+2 & a+10 \end{array}\right]$$ *Ensure increasing zeros in bottom left.

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 6+a & 2 \\ 0 & 1 & a^2 & a+4 \end{array}\right] \begin{array}{l} R_1 \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$ *Write all row operations beside matrix on each step.

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 6+a & 2 \\ 0 & 0 & a^2-a-6 & a+2 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 - R_2 \end{array}$$

(Q1a) Find a that gives no solutions:
Last row in form of: $[0\ \ 0\ \ 0\ \ |\ \ B]\ B \neq 0$
$\therefore a^2 - a - 6 = 0$ and $a + 2 \neq 0$
Solving to get $a = 3, -2$ and $a \neq -2 \therefore a = 3$

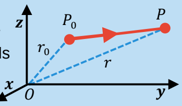
(Q1b) Find a that gives infinite solutions:
Last row in form of: $[0\ \ 0\ \ 0\ \ |\ \ 0]$
$\therefore a^2 - a - 6 = 0$ and $a + 2 = 0$
Solving to get $a = 3, -2$ and $a = -2 \therefore a = -2$

(Q1c) Find a that gives a unique solution:
Last row in form of: $[0\ \ 0\ \ A\ \ |\ \ B]\ A, B \neq 0$
$\therefore a^2 - a - 6 \neq 0$ and $a + 2 \neq 0$
Solving to get $a \neq 3, -2$ and $a \neq -2$
$\therefore a \neq -2$ has unique solution $(a \in \mathbb{R}: a \neq -2)$

TYPES OF LINES AND PLANES

Line Definition and Equations

- Lines contain a series of collinear points that extends infinitely in both directions.



- Parametric equation of a line:

$x = a + \lambda d,\ y = b + \lambda e,\ z = c + \lambda f$

- (a, b, c) : is r_0 (*i.e.* point on the line).
- (d, e, f) : is $r - r_0$ (*i.e.* vector direction).
- λ : magnitude/direction constant.

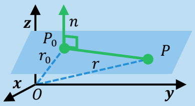
- Cartesian equation of a line:

$\frac{x-a}{d} = \frac{y-b}{e} = \frac{z-c}{f}$

- (a, b, c) : is r_0 (*i.e.* vector origin location).
- (d, e, f) : is $r - r_0$ (*i.e.* vector direction).

Plane Definition and Equations

- Planes extend infinitely in all directions and has no thickness.



- Vector equation of a plane:

$(r - r_0).n = 0$ | $r.n = r_0.n$ | $r.n = c$

- P and P_0 : two points on the plane.
- n : normal (perpendicular) to the plane.

- Cartesian equation of a plane:

$A(x - x_0) + B(y - y_0) + C(z - z_0) = D$
$Ax + By + Cz + D = 0$

- (A, B, C) : normal vector to the plane.
- (x_0, y_0, z_0) : point on the plane.

VECTOR RULES

Vector Rules and Notation

- Given $\tilde{a} = (x_a, y_a, z_a)$ and $\tilde{b} = (x_b, y_b, z_b)$:

$\overrightarrow{AB} = \tilde{b} - \tilde{a}$ | $|x| = \sqrt{x^2 + y^2 + z^2}$
$|\overrightarrow{AB}| = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2}$

Unit Vector ($\hat{x}$)

- Returns a vector with the same direction as vector x but with a magnitude of 1.

$\hat{x} = a/|a|$ | $|\hat{a}| = 1$

Dot Product ($a.b$)

- Returns scalar result (a single number).

$a.b = (x_a \times x_b) + (y_a \times y_b) + (z_a \times z_b)$
$a.b = |a||b|cos\theta$ | $a.a = |a|^2$
a and b are perpendicular if $a.b = 0$

Cross Product ($a \times b$)

- Returns vector result (vector with co-ords).
- Returns a vector normal to a plane.

$a \times b = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} \times \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = \begin{pmatrix} y_a z_b - z_a y_b \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{pmatrix}$
$x \times y = \hat{n}|x||y|sin\theta$

- $\hat{n}$: unit vector perpendicular to x and y.

Vector Equations of a Sphere

- Vector equation of a sphere:

$|r - c| = a$

- c : co-ords of the centre of sphere.
- a : radius of the sphere.

- Cartesian equation of a sphere:

$(x-a)^2 + (y-b)^2 + (z-c)^2 = r$

- (a, b, c) : co-ords of the centre of sphere.
- a : radius of the sphere.

APPLICATIONS OF LINES

Application of Line Vectors Examples

- Finding the vector equation of a line:

(Q1) Co-ords of $A(2,1,-3)$ and $B(4,5,-1)$.
$\overrightarrow{AB} = \tilde{b} - \tilde{a} = 2i + 4j + 2k$ and hence,
$r = (2i + j - 3k) + \lambda(2i + 4j + 2k)$

- Finding the parametric equation of a line:

(Q2) Point is $A(-7,2,4)$ and parallel to the line given by $x = 5 - 8t, y = 6 + t, z = -12t$
$x = -7 - 8t, y = 2 + t, z = 4 - 12t$ and then solve for t: $(-7 - x)/8 = y - 2 = (4 - z)/12$

- Test if a point is perpendicular to a line:

(Q3) Point is $A(1,2,1)$ and the equation of the line is $r = (i + 2j + 3k) + \lambda(4i + 2j - 8k)$
$(i + 2j + k).(4i + 2j - 8k) = 4 + 4 - 8 = 0$
Hence, the point is perpendicular to the line.

- Intersection of two moving vectors:

(Q4) Find intersection points between lines $A = (-7i + 9j - 5k) + \lambda(5i - 4j + 2k)$ and $B = (-6i - 5j + 2k) + \mu(9i + 6j - 3k)$
Solve the i, j and k parts for λ and μ:
$-7 + 5\lambda = -6 + 9\mu$, $9 - 4\lambda = -5 + 6\mu$ and $-5 + 2\lambda = 2 - 3\mu$ and hence, $\lambda = 2$, $\mu = 1$ therefore point of intersection is $(3, 1, -1)$

► Topic Is Continued In Next Column ◄

APPLICATIONS OF LINES

Application of Line Vectors Examples

- Collision of two moving vectors:

(Q5) $A = (2i + 1j - 3k) + \lambda(7i + 10j - 3k)$ and $B = (5i + 28j - 6k) + \mu(6i + j - 2k)$ where velocity is measured in km/h. Find collision:

- Equating i coefficients: $2 + 7\lambda = 5 + 6\mu$
- Equating j coefficients: $1 + 10\lambda = 28 + 1\mu$
- Equating k coefficients: $-3 - 3\lambda = -6 - 2\mu$

Solving the first two equations for λ and μ: $\lambda = 3$ and $\mu = 3$. Substitute into third equation (k coefficient): $-3 - 3(3) = -6 - 2(3) \to 6 = 6$ which is consistent, so a collision occurs as times λ and μ are the same (@ $t = 3$). Finding collision point, substitute $t = 3$ back into A or B:
$A = (2i + 1j - 3k) + 3(7i + 10j - 3k)$
$\therefore$ A and B collide at $(23i + 31j - 12k)$

- Shortest distance between two vectors:

(Q6) $A = (2i + j - 3k) + \lambda(7i + 10j - 3k)$ and $B = (-5i + 20j + k) + \mu(-3i - j + 7k)$ where velocity is measured in km/h.

$\vec{d} = \overrightarrow{BA} + ({}_AV_B)t$ | $\vec{d}\ .\ {}_AV_B = 0$

- $\vec{d}$: shortest distance between A and B.
- $\overrightarrow{BA} = \tilde{a} - \tilde{b}$: vector between A and B.
- ${}_AV_B = V_A - V_B$: relative velocity B to A.

$\overrightarrow{BA} = \begin{bmatrix} 7 \\ -19 \\ -4 \end{bmatrix}$ and ${}_AV_B = \begin{bmatrix} 7 \\ 10 \\ -3 \end{bmatrix} - \begin{bmatrix} -3 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \\ -10 \end{bmatrix}$
$\vec{d} = \overrightarrow{BA} + ({}_AV_B)t = (7, -19, -4) + t(10, 11, -10)$

- Using ClassPad to find time,

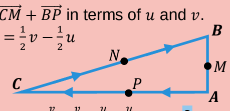
$= dotP\left(\begin{bmatrix} 7 \\ -19 \\ -4 \end{bmatrix} + t\begin{bmatrix} 10 \\ 11 \\ -10 \end{bmatrix}, \begin{bmatrix} 10 \\ 11 \\ -10 \end{bmatrix}\right) = 0.308\ hr$

- Using ClassPad to find shortest distance,

$= |(7, -19, -4) + 0.308(10, 11, -10)| = 19.9km$

- Vector proofs:

(Q7) Triangle ABC i with the midpoints of each side M, N and P shown. Let $\overrightarrow{AC} = u$ and $\overrightarrow{CB} = v$. Express $\overrightarrow{AN} + \overrightarrow{CM} + \overrightarrow{BP}$ in terms of u and v.
$\overrightarrow{CM} = \frac{1}{2}(v - u) = \frac{1}{2}v - \frac{1}{2}u$
$\overrightarrow{AN} = u + \frac{1}{2}v$
$\overrightarrow{BP} = -\frac{1}{2}u - v$



$\overrightarrow{AN} + \overrightarrow{CM} + \overrightarrow{BP} = u + \frac{v}{2} + \frac{v}{2} - \frac{u}{2} - \frac{u}{2} - v = 0$

APPLICATIONS OF PLANES

Application of Plane Vectors Examples

- Vector equation of a plane:

(Q1) A plane contains the point $(5, -7, 2)$ and has a normal parallel to $(3, 0, -1)$
$\begin{bmatrix} x-5 \\ y+7 \\ z-2 \end{bmatrix} . \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 0$ and hence, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} . \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 13$

- Cartesian equation of a plane

(Q2) Find cartesian equation of $r.[3, -6, 9] = 36$
Let $r = (x, y, z) \therefore (x, y, z).(3, -6, 9) = 36$
$\therefore 3x - 6y + 9z = 36 \to x - 2y + 3z = 12$

- Equation of plane passing through 3 points:

(Q3) Find the equation of a plane that passes through points $A(1,1,1)$, $B(-1,1,0)$ and $C(2,0,3)$
$\overrightarrow{AB} = (-2, 0, -1)$ and $\overrightarrow{AC} = (1, -1, 2)$
$\overrightarrow{AB} \times \overrightarrow{AC} = (-1, 3, 2)$ and hence equation of the plane is $-x + 3y + 2z + D = 0$. Sub any point to find D: $-(2) + 3(0) + 2(3) + D = 0$
$D = -4$ hence $-x + 3y + 2z - 4 = 0$

- Plane with a point and orthogonal to line:

(Q4) Find plane that passes through $A(3, 0, -4)$ and orthogonal to $r(t) = \langle 12 - t, 1 + 8t, 4 + 6t \rangle$
$n = (-1, 8, 6) \therefore$ have normal and point in plane:
$-(x - 3) + 8(y - 0) + 6(z + 4) = 0$
Expand and rearrange: $-x + 8y + 6z = -27$

- Finding where a line intersects with a plane:

(Q5) A plane contains point $(5, -7, 2)$ and has a normal vector parallel to $(3, 0, -1)$, where does it intersect $A = (-10i + 4j - 9k) + \lambda(2i + j - 6k)$
$r.n = r_0.n \to \begin{bmatrix} -10+2\lambda \\ 4+\lambda \\ -9-6\lambda \end{bmatrix} . \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} . \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$
$\therefore r.n = c \to \begin{bmatrix} -10+2\lambda \\ 4+\lambda \\ -9-6\lambda \end{bmatrix} . \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 13$ *Solve for λ:
$\lambda = 17/6$ and sub into $A = (-\frac{26}{6}, \frac{41}{6}, -26)$

- Shortest distance from point to a plane:

(Q6) Find the shortest distance between point $(4, -4, 3)$ and the plane $2x - 2y + 5z + 8 = 0$

$\vec{d} = \frac{absolute\ value(Ax + By + Cz + D)}{\sqrt{A^2 + B^2 + C^2}}$

- $\vec{d}$: shortest distance between point/plane.
- (A, B, C) : normal vector to the plane.
- (x, y, z) : point outside of the plane.

$\vec{d} = \frac{abs[(2 \times 4) + (-2 \times -4) + (3 \times 5) + 8]}{\sqrt{2^2 + (-2)^2 + 5^2}}$
$\therefore \vec{d} = 39/\sqrt{33} = 6.79$ units.

APPLICATIONS OF SPHERES

Application of Vector Sphere Examples

(Q1) Find the radius and co-ordinates of the centre of the sphere with the equation:
$x^2+y^2+z^2+2x+4y-6z-50=0$

Rearrange: $x^2+y^2+z^2+2x+4y-6z=50$

$\therefore a=2, b=4, c=-6, d=50$

$LHS=\left(x+\frac{a}{2}\right)^2+\left(y+\frac{b}{2}\right)^2+\left(z+\frac{c}{2}\right)^2$

$LHS=(x+1)^2+(y+2)^2+(z-3)^2$

$RHS=d+\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2+\left(\frac{c}{2}\right)^2$ *Expand and simplify

$RHS=50+1+4+9=64=8^2$

Hence, centre at $(-1,-2,3)$ and radius of 8.

(Q2) Find vector equation of sphere with a diameter AB where $A(-1,0,6)$ and $B(3,6,18)$:

$Centre=\left(\frac{-1+3}{2},\frac{0+6}{2},\frac{6+18}{2}\right)=(1,3,12)$

$Radius=(|1--1{,}3-0{,}12-6|)=(|2{,}3{,}6|)$

$=\sqrt{2^2+3^2+6^2}=7$ then sub into equation:

$|r-c|=a\rightarrow |r-(1,3,12)|=7$

TRIGONOMETRY

TRIGONOMETRIC FORMULAE

Exact Values of Trigonometric Ratios

Deg.	0°	30°	45°	60°	90°
Rad.	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
Sin	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
Cos	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
Tan	0	$\sqrt{3}/3$	1	$\sqrt{3}$	N/A

Trigonometric Identities

- Sum and difference identities:

$\sin(a\pm b)=\sin(a)\cos(b)\pm\sin(b)\cos(a)$

$\cos(a\pm b)=\cos(a)\cos(b)\mp\sin(a)\sin(b)$

$\tan(a\pm b)=\frac{\tan(a)\pm\tan(b)}{1\mp\tan(a)\tan(b)}$

- Reciprocal identities:

$cosec(x)=\frac{1}{\sin(x)}$ | $\sec(x)=\frac{1}{\cos(x)}$ | $\cot(x)=\frac{1}{\tan(x)}$

- Pythagorean identities:

$\sin^2\theta+\cos^2\theta=1$ | $1+\tan^2\theta=\sec^2\theta$

- Quotient identities:

$\tan(x)=\frac{\sin(x)}{\cos(x)}$ | $\cot(x)=\frac{\cos(x)}{\sin(x)}$

- Co-function identities:

$\sin\left(\frac{\pi}{2}-x\right)=\cos(x)$ | $\cos\left(\frac{\pi}{2}-x\right)=\sin(x)$

- Parity identities (i.e. even and odd):

$\sin(-x)=-\sin(x)$ | $\cos(-x)=\cos(x)$

$\tan(-x)=-\tan(x)$ | $\sec(-x)=\sec(x)$

- Double angle identities:

$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1=1-2\sin^2(x)$

$\sin(2x)=2\sin(x)\cos(x)$

$\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}$

- Combination angle identities:

$cosXcosY=\frac{1}{2}(cos(X-Y)+cos(X+Y))$

$sinXsinY=\frac{1}{2}(cos(X-Y)-cos(X+Y))$

$sinXcosY=\frac{1}{2}(sin(X+Y)+sin(X-Y))$

- Power reducing identities:

$\sin^2(x)=\frac{1-\cos(2x)}{2}$ | $\cos^2(x)=\frac{1+\cos(2x)}{2}$

- Limits of sine and cosine:

$\lim_{x\to 0}\frac{\sin(x)}{x}=1$ | $\lim_{x\to 0}\frac{1-\cos(x)}{x}=0$

Triangle Laws

- Sine Rule (i.e. finding angles and sides)

$\frac{a}{sinA}=\frac{b}{sinB}$ | $\frac{sinA}{a}=\frac{sinB}{b}$

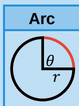
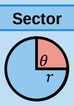
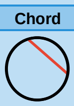
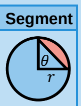
- Cosine Rule (i.e. finding angles and sides)

$c^2=a^2+b^2-2\times a\times b\times cos(C)$

$Angle\ C=cos^{-1}\left(\frac{a^2+b^2-c^2}{2\times a\times b}\right)$

Circle Measure

- Common circle measure terminology:

Arc	Sector	Chord	Segment
			

- Circle measure formulae:

Length Arc	Area Sector	Area Segment
$r\theta$	$\frac{1}{2}r^2\theta$	$\frac{1}{2}r^2(\theta-sin\theta)$

CALCULUS

DIFFERENTIATION RULES

Derivative Laws

Type	Equation	1st Derivative
Product Rule	$y=uv$	$\frac{dy}{dx}=u'v+uv'$
Quotient Rule	$y=\frac{u}{v}$	$\frac{dy}{dx}=\frac{u'v-uv'}{v^2}$
Chain Rule	$y=[f(x)]^n$	$\frac{dy}{dx}=n[f(x)]^{n-1}\times f'(x)$
Chain Leibniz	$x=f(t)$, $y=f(t)$	$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dx}{dt}$

Common Functions and Derivatives

Function	Equation	1st Derivative
Polynomial	$y=ax^n$	$\frac{dy}{dx}=n\times ax^{n-1}$
Exponential (Euler)	$y=e^{f(x)}$	$\frac{dy}{dx}=f'(x)\times e^{f(x)}$
Reciprocal	$y=\frac{1}{x}=x^{-1}$	$\frac{dy}{dx}=\frac{-1}{x^2}=-x^{-2}$
Sine	$y=\pm sin(x)$	$\frac{dy}{dx}=\pm cos(x)$
Cosine	$y=\pm cos(x)$	$\frac{dy}{dx}=\mp sin(x)$
Tangent	$y=\pm tan(x)$	$\frac{dy}{dx}=\pm\sec^2(x)$
Natural Logarithm	$y=ln[f(x)]$	$\frac{dy}{dx}=\frac{f'(x)}{f(x)}$
Exponential (Non-Euler)	$y=a^x$	$\frac{dy}{dx}=\ln(a)\times a^x$

INTEGRAL LAWS

Integration Laws

$\int_a^b f(x)=-\int_b^a f(x)$ | $\int_a^a f(x)=0$

$\int a\times f(x)\,dx=a\times\int f(x)\,dx$

$\int[f(x)\pm g(x)]dx=\int f(x)dx\pm\int g(x)dx$

$\int_b^a f(x)dx+\int_c^b f(x)dx=\int_c^a f(x)dx$

Common Functions and Integrals

Function	Equation	Integral
Polynomial	$\int x^n dx$	$\frac{x^{n+1}}{n+1}+c$
Chain Rule	$\int f'(x)[f(x)]^n\,dx$	$\frac{[f(x)]^{n+1}}{n+1}+c$
Exponential (Euler)	$\int e^{f(x)}\,dx$	$\frac{e^{f(x)}}{f'(x)}+c$
Reciprocal	$\int\frac{f'(x)}{f(x)}\,dx$	$ln\|f(x)\|+c$
Sine	$\int sin(x)\,dx$	$-cos(x)+c$
Cosine	$\int cos(x)\,dx$	$sin(x)+c$
Secant	$\int sec^2(x)\,dx$	$tan(x)+c$

Fundamental Theorem of Calculus

$\frac{d}{dx}\left(\int_a^x f(t)dt\right)=f(x)$ | $\int_a^b f(x)\,dx=F(b)-F(a)$

Integration by Parts

$\int uv'\,dx=uv-\int u'v\,dx$

Area Between Curves Formulae

- Upper and Lower Bounds on the x-axis:

$\int_a^b\left(\begin{matrix}upper\\function\end{matrix}\right)-\left(\begin{matrix}lower\\function\end{matrix}\right)dx$

- Upper and Lower Bounds on the y-axis:
  - x is the subject of equation in terms of y.

$\int_c^d\left(\begin{matrix}right\\function\end{matrix}\right)-\left(\begin{matrix}left\\function\end{matrix}\right)dy$

INTEGRATION TECHNIQUES

Integration Examples

(Q1) If $u=ln\sqrt{x+1}$, determine $\int\frac{ln\sqrt{x+1}}{2x+2}dx$

- Unpacking substitution:

$u=ln\sqrt{x+1}$ $\quad\frac{du}{dx}=\frac{1}{2(x+1)}$ $\quad$ Denominator:

$e^u=\sqrt{x+1}$ $\quad$ $2x+2$

$e^{2u}=x+1$ $\quad dx=2(x+1)du$ $\quad=2e^{2u}$

$x=e^{2u}-1$ $\quad dx=2e^{2u}du$

- Substituting u into integral:

$\int\frac{(u)(2e^{2u})}{2e^{2u}}du=\int u\,du=\frac{u^2}{2}=\frac{(ln\sqrt{x+1})^2}{2}+c$

(Q2) Find the integral $\int\sin^5(2x)dx$

$=\int sin2x(\sin^2 2x)^2dx$ *Expand and simplify

$=\int sin2x(1-\cos^2 2x)^2dx$

$=\int sin2x(1-2\cos^2 2x+\cos^4 2x)dx$

$=\int sin2x-2sin2xcos^2 2x+sin2xcos^4 2xdx$

$=-\frac{cos2x}{2}+\frac{\cos^3 2x}{3}-\frac{\cos^5 2x}{10}+c$

► Topic Is Continued In Next Column ◄

INTEGRATION TECHNIQUES

Integration Examples

(Q3) Let $x=2sin\theta$, determine $\int_0^2\sqrt{4-x^2}\,dx$

- Unpacking substitution:

$dx=2cos\theta d\theta$ $\quad$ Integral $\sqrt{4-x^2}$

$\theta=sin^{-1}(x/2)$ $\quad=\sqrt{4-(2cos\theta)^2}$

When $x=2$: $\quad=\sqrt{4-4\cos^2\theta}$

$\theta=sin^{-1}(2/2)=\pi/2$ $\quad=2\sqrt{1-\cos^2\theta}$

When $x=0$: $\quad=2sin\theta$

$\theta=sin^{-1}(0/2)=0$ $\quad$ *Substitute limits also

- Substituting u into integral:

$\int_0^2\sqrt{4-x^2}\,dx=\int_0^{\frac{\pi}{2}}2cos\theta\times 2cos\theta\,d\theta$

$=\int_0^{\frac{\pi}{2}}4\cos^2\theta\,d\theta=\frac{4}{2}\int_0^{\frac{\pi}{2}}(1+cos2\theta)d\theta$

$=2\left[\theta+\frac{sin2\theta}{2}\right]_0^{\frac{\pi}{2}}=2\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right]=\pi$

(Q4) Determine $\int(3x+11)/(x^2-x-6)\,dx$

- Finding partial fractions (i.e. on page 2):

$=\int\frac{4}{x-3}-\frac{1}{x+2}dx=4ln|x-3|-ln|x+2|+c$

IMPLICIT DIFFERENTIATION

Implicit Differentiation Rules

- Used for functions that in terms of x and y.
- Chain rule to differentiate y with respect to x:

$d/dx=d/dy\times dy/dx$

$\frac{d}{dx}y^2=\frac{d}{dy}y^2\times\frac{dy}{dx}=2y\times\frac{dy}{dx}$

- Common implicit derivatives of y:

$y\rightarrow\frac{dy}{dx}$ | $y^2\rightarrow 2y\frac{dy}{dx}$ | $xy\rightarrow y+x\frac{dy}{dx}$

- Method of finding implicit derivatives:

Step 1	Differentiate both sides of the equation with respect to x.
Step 2	Collect all terms containing dy/dx on one side of the equation
Step 3	Factor out dy/dx and solve for dy/dx (i.e. by dividing both sides).

Implicit Differentiation Examples

(Q1) Determine derivative of $y=sinx+cosy$

$\frac{dy}{dx}=cosx-siny\frac{dy}{dx}$ $\qquad\frac{dy}{dx}(1+siny)=cosx$

$\frac{dy}{dx}+siny\frac{dy}{dx}=cosx$ $\qquad\frac{dy}{dx}=\frac{cosx}{1+siny}$

(Q2) Find gradient at $(2,-1)$ of $x+x^2y^3=-2$

$1+2xy^3+x^23y^2\frac{dy}{dx}=0\rightarrow\frac{dy}{dx}=\frac{-1-2xy^3}{x^23y^2}$

Sub $x=2,y=-1\rightarrow\frac{dy}{dx}=\frac{-1-2\times2\times(-1)^3}{2^2\times3\times(-1)^2}=\frac{3}{12}=\frac{1}{4}$

Q3) Find co-ords of points where tangent to the curve $x^2+2xy+3y^2=18$ is horizontal.

$2x+2y+2x\frac{dy}{dx}+6y\frac{dy}{dx}=0$

$\frac{dy}{dx}(2x+6y)=-2x-2y\rightarrow\frac{dy}{dx}=-\left(\frac{x+y}{x+3y}\right)$

Solve for when $\frac{dy}{dx}=0$ hence $x=-y$

Substitute into original: $y^2-2y^2+3y^2=18$

$y^2=9$ and hence, $(3,-3)$ and $(-3,3)$

(Q4) Point (a,b) lies on the curves $x^2-y^2=5$ and $xy=6$. Prove that the tangents of both of these curves at point (a,b) are perpendicular.

- Differentiating x^2-y^2 with respect to x:

$x^2-y^2=5\rightarrow 2x-2y\frac{dy}{dx}=0\rightarrow\frac{dy}{dx}=\frac{x}{y}$

At point (a,b) the slope is $m_1=x/y$

- Differentiating $xy=6$ with respect to x:

$xy=6\rightarrow y+x\frac{dy}{dx}=0\rightarrow\frac{dy}{dx}=-\frac{y}{x}$

At point (a,b) the slope is $m_2=-y/x$

- Lines are perpendicular if $m_1\times m_2=-1$

$m_1\times m_2=(x/y)\times(-y/x)=-1$ hence **yes**.

DIFFERENTIAL EQUATIONS

Solving by Separation of Variables

Step 1	Move all y terms (including dy) to one side of the equation and all x terms (including dx) to the other.
Step 2	Integrate one side with respect to y and the other with respect to x. Add a "+c" to end of solution.
Step 3	Simplify and solve for c if given set of co-ords from original function.

Solving Differential Equation Examples

(Q1) Find equation of circle passing through $(2,4)$ with a gradient $dy/dx=1/y-x/y$

$\frac{dy}{dx}=\frac{1-x}{y}\rightarrow\int y\,dy=\int 1-x\,dy$

$\frac{y^2}{2}=x-\frac{x^2}{2}+k\rightarrow y^2=2x-x^2+c$

- Apply condition $(2,4)$ to solve for c:

$4^2=2(2)-2^2+c\rightarrow 16=4-4+c\rightarrow c=16$

$\therefore y^2=2x-x^2+16\rightarrow y^2+x^2-2x=16$

(Q2) Find general solution for the differential equation $y'=6y^2x$ given that $x=1, y=1/25$

$\frac{dy}{dx}=6y^2x\rightarrow\int\frac{dy}{y^2}=\int 6xdx\rightarrow-\frac{1}{y}=3x^2+c$

- Apply condition $(1,1/25)$ to solve for c:

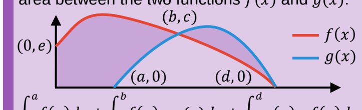
$-25=3+c\rightarrow c=-28$

$\therefore-\frac{1}{y}=3x^2-28$ $\qquad\therefore y=\frac{1}{28-3x^2}$

DIFFERENTIATION RULES

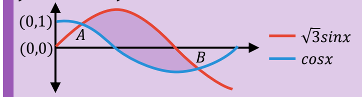
Area Between Curves Examples

(Q1) Find an expression for finding the shaded area between the two functions $f(x)$ and $g(x)$:



$\int_0^a f(x)dx+\int_a^b f(x)-g(x)dx+\int_b^d g(x)-f(x)dx$

(Q2) Determine the area between the two curves $y=cosx$ and $y=\sqrt{3}sinx$ as shown below:



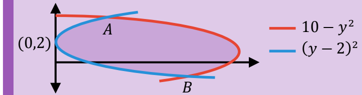
- Finding intersection points A and B:

$\frac{1}{\sqrt{3}}=\frac{sinx}{cosx}\rightarrow\frac{\sqrt{3}}{3}=tanx\rightarrow x=\frac{\pi}{6},\frac{7\pi}{6}...$

- Finding area between the two curves:

$\int_{\frac{\pi}{6}}^{\frac{7\pi}{6}}\sqrt{3}sinx-cosx\,dx=\left[-\sqrt{3}cosx-sinx\right]_{\frac{\pi}{6}}^{\frac{7\pi}{6}}=4$

(Q3) Determine the area between the two curves $x=10-y^2$ and $x=(y-2)^2$ as shown below:



- Finding intersection points A and B:

$10-y^2=(y-2)^2\rightarrow 10-y^2=y^2-4y+4$

$0=2y^2-4y+6=(y-3)(y+1)\therefore y=-1,3$

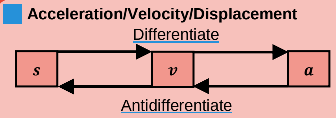
- Finding area between the two curves:

$\int_{-1}^3 right-left\,dy=\int_{-1}^3 10-y^2-(2-y)^2\,dy$

$=\left[10y-\frac{1}{3}y^3-\frac{1}{3}(2-y)^3\right]_{-1}^3=21\frac{1}{3}$

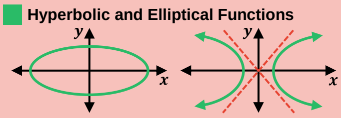
VECTOR CALCULUS

Acceleration/Velocity/Displacement



Δ Displacement	Distance Travelled
$Change=\int_a^b v(t)\,dt$	$Total=\int_a^b \|v(t)\|\,dt$

Hyperbolic and Elliptical Functions



- Equation and features of an ellipse:

$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

- $2a$: width of the ellipse (on the x-axis).
- $2b$: height of the ellipse (on the y-axis).
- (h,k) : co-ords of centre of ellipse.

- Equation and features of a hyperbola:

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

- $y=k\pm\frac{b}{a}(x-h)$: hyperbola asymptotes.
- (h,k) : co-ords of centre of hyperbola.

Vector Calculus Examples

(Q1) Velocity of a golf ball at $t=0$ from origin is given by $v=35i+5j+20k$ measured in m/s. Note: i is unit vector for movement in direction of the hole, j is movement perpendicular to i and k is unit vector for vertical movement.

(Q1a) If $a=-9.8k$, find displacement vector:

$v=35i+5j+(20-9.8t)k$

$s=\int v\,dt=35ti+5tj+(20t-4.9t^2)$

(Q1b) How long does the ball spend in the air?

Solve $a(t)=0\rightarrow 20t-4.9t^2=0\rightarrow t=4.08\ s$

(Q1c) What is ball speed when it hits ground?

$|v(4.08)|=\sqrt{35^2+5^2+(-20)^2}=40.62\ m/s$

(Q1d) The hole is 150 metres away from tee off. How far is the ball when it hits the ground?

$r(4.08)=142.8i+20.4j$ $\quad Dist=|7.2i+20.4j|$

$150i-(142.8i+20.4j)$ $\quad\sqrt{7.2^2+20.4^2}=22\ m$

(Q2) Find cartesian equation of the particle that moves according to $v=(3cost)i+(sint)j$

$x=3cost\rightarrow cost=x/3$ and $y=sint$

$sin^2t+cos^2t=y^2+(x/3)^2=1\rightarrow\frac{x^2}{9}+y^2=1$

(Q3) Find cartesian equation of the particle that moves according to $v=(3tant)i+(4sect)j$

$x=3tant\rightarrow tant=x/3$ and $sect=y/4$

$1+\tan^2\theta=\sec^2\theta\rightarrow 1+(x/3)^2=(y/4)^2$

Expand and simplify: $1+\frac{x^2}{9}=\frac{y^2}{16}\rightarrow\frac{y^2}{16}-\frac{x^2}{9}=1$

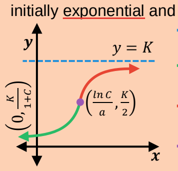
ATAR Math Specialist
Units 3 & 4 Exam Notes

Created by Anthony Bochrinis

LOGISTIC FUNCTION

Logistic Function Notation and Graph

- Model to predict population growth that is initially exponential and then slows down.



$\frac{dP}{dt} = aP\left(1 - \frac{P}{K}\right)$	$P = \frac{K}{1 + Ce^{-at}}$

- P : population at time $= t$.
- K : carrying capacity (*i.e.* maximum pop).
- a : growth rate ("$-a$" makes it decay).
- C : constant (specific to the question).

Logistic Function Examples

(Q1) Population of fish in a lake t years after 2000 is modelled by the function: $P = \frac{500}{1 + 9e^{-0.07t}}$

(Q1a) What is the population in year 2010?

$P = 500/(1 + 8e^{-0.07 \times 10}) = 91.41 \approx$ **91**

(Q1b) What is the carrying capacity of fish?

As $t \to \infty$, $P \to 500/(1 + 9e^{-\infty}) \to K =$ **500**

(Q1c) At what time is there maximum growth?

Maximum growth occurs when population is equal to $K/2 = 250$ fish. At this point, the time is equal to $\ln C / a = \ln 9 / 0.07 =$ **31.39** years.

(Q1d) Find the derivative of the function:

$\frac{dP}{dt} = 0.07P\left(1 - \frac{P}{500}\right) = \frac{7P}{100} - \frac{7P^2}{50000}$

(Q1e) Derive original function from derivative, given the initial condition $P(0) = 50$:

Step 1	Combine into one fraction and integrate by separating variables.
Step 2	Split the large fraction by using partial fractions to integrate.
Step 3	Use log law $\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$ to simplify and then sub $e^c = C$
Step 4	Solve for C by subbing and solving using the initial condition (t, P).
Step 5	Rearrange the equation so that the population P is the subject.

- Combine fractions and separate variables:

$\frac{dP}{dt} = \frac{7}{100}\left(\frac{500P - P^2}{500}\right) \to \int \frac{500}{500P - P^2} dP = \int \frac{7}{100} dt$

- Use partial fractions to integrate LHS:

$\frac{500}{500P - P^2} = \frac{500}{P(500-P)} = \frac{A}{P} + \frac{B}{500-P} = \frac{500A - AP + BP}{P(500-P)}$

$\therefore 500A = 500 \;\&\; -AP + BP = 0 \therefore A = 1, B = 1$

$\therefore \int \frac{1}{P} + \frac{1}{(500-P)} dP = \int \frac{7}{100} dt$ *Integrate using integral laws

$\ln|P| - \ln|500 - P| = \frac{7t}{100} + c$

- Use log laws and $e^c = C$ substitution:

$\ln\left|\frac{P}{500-P}\right| = 0.07t + c \quad \frac{P}{500-P} = e^c \times e^{0.07t}$

$\frac{P}{500-P} = e^{0.07t+c} \quad \frac{P}{500-P} = Ce^{0.07t}$

- Solve C using initial condition $P(0) = 50$:

$\frac{50}{500-50} = Ce^{0.07(0)} \to \frac{50}{450} = Ce^0 \to C = \frac{1}{9}$

- Rearrange to make P the subject:

$\frac{P}{500-P} = \frac{1}{9}e^{0.07t} \quad 9P + e^{0.07t}P = 500e^{0.07t}$

$9P = (500 - P)e^{0.07t} \quad P(9 + e^{0.07t}) = 500e^{0.07t}$

- Divide both sides of equation by $e^{0.07t}$:

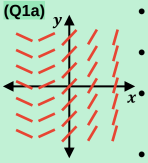
$P\left(\frac{9}{e^{0.07t}} + 1\right) = 500 \to P = \frac{500}{1 + 9e^{-0.07t}}$

SLOPE FIELDS

Slope Field Examples

(Q1) Find a general differential equation for the slope fields below and explain your reasoning.

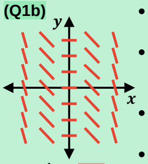
(Q1a)



- Quadratic equation formed by isoclines.
- Convex nature, ∴ a has a positive value.
- x-intercept on the negative x-axis, hence b has a positive value.
- dy/dx has format of a linear equation.

$\frac{dy}{dx} = ax + b$

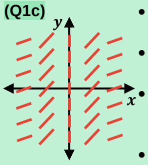
(Q1b)



- Pattern of the isoclines forms a cubic equation.
- Isoclines have consistent negative gradient, ∴ a has a negative value.
- No other inflection points ∴ bx has value of 0.
- dy/dx has format of a quadratic equation.

$\frac{dy}{dx} = ax^2$

(Q1c)



- Pattern of the isoclines form hyperbolic function.
- Gradient is ∞ at $x = 0$ ∴ vertical asymptote.
- Isoclines have consistent positive gradient, ∴ a has a positive value.
- dy/dx has format of derivative of hyperbolic function.

$\frac{dy}{dx} = \frac{a}{x^2} + b$

SIMPLE HARMONIC MOTION

Period, Amplitude and Phase

- Changing values of $af[b(x + c)] + d$:
- Period: how long it takes for a trigonometric function to complete 1 full cycle.
  - Period relates to 'b' in each equation:

Ratio	Sine	Cosine	Tangent
Period	2π	2π	π
b	$2\pi/Period$	$2\pi/Period$	$\pi/Period$

Ratio	Cosecant	Secant	Cotangent
Period	2π	2π	π
b	$2\pi/Period$	$2\pi/Period$	$\pi/Period$

- Amplitude: maximum vertical distance in units from the x-axis to max/min points.
  - Amplitude relates to 'a' in each equation:

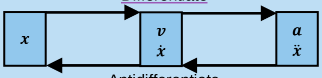
$$a = \frac{max\, y_{value} - min\, y_{value}}{2}$$

- Phase: refers to any left or rightward shifts.
  - Phase relates to 'c' in each equation.
- Vertical Shift: relates to 'd' in each equation.

Simple Harmonic Motion Rules (*SMH*)

- Explores motion with variable acceleration (*i.e.* moves according to a trig function).
- Displacement/velocity/acceleration notation:

Differentiate



Antidifferentiate

- Finding acceleration of SMH:

$$\ddot{x} = a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

- Proofs that object undergoes SMH:

$a = \frac{d^2x}{dt^2} = -n^2x$	$v^2 = n^2(a^2 - x^2)$

- n : value of b in $af[b(x + c)] + d$, also known as $2\pi/Period$ or $\pi/Period$ depending on the trigonometric function.
- a : amplitude of the motion.

Simple Harmonic Motion Examples

(Q1) Particle accelerates with SMH according to $a = 8\cos 2t$. Also, initially the particle is stationary and at time $t = 0$, $x = 3$ metres.

(Q1a) Find velocity & displacement functions:

$v = \int a(t)\, dt = \int 8\cos 2t\, dt = 4\sin 2t + c$

At $t = 0, v = 0 \;\therefore c = 0 \;\therefore v =$ **$4\sin 2t$**

$x = \int v(t)\, dt = \int 4\sin 2t\, dt = -2\cos 2t + c$

At $t = 0, x = 2 \;\therefore c = 4 \;\therefore x =$ **$-2\cos 2t + 4$**

(Q1b) Find particle speed at $x = 0.75$ metres:

$v^2 = k^2(A^2 - x^2) = 2^2(2^2 - 0.75^2) =$ **$13.8 m/s$**

(Q1c) Find distance travelled after 3 seconds:

$Dist = \int_0^3 |v(t)|\, dt = \int_0^3 |4\sin 2t|\, dt =$ **$7.92m$**

(Q2) Particle is moving in a line with distance from origin given by $x = 2\cos\left(\frac{\pi t}{3}\right) - 3\sin\left(\frac{\pi t}{3}\right)$.

(Q2a) Prove that particle is undergoing SMH:

$\dot{x} = -2\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi t}{3}\right) - 3\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi t}{3}\right)$

$\ddot{x} = -2\left(\frac{\pi}{3}\right)\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi t}{3}\right) + 3\left(\frac{\pi}{3}\right)\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi t}{3}\right)$

$\ddot{x} = \left(\frac{-2\pi^2}{9}\right)\cos\left(\frac{\pi t}{3}\right) + \left(\frac{3\pi^2}{9}\right)\sin\left(\frac{\pi t}{3}\right)$

$\ddot{x} = \frac{\pi^2}{9}\left(-2\cos\left(\frac{\pi t}{3}\right) + 3\sin\left(\frac{\pi t}{3}\right)\right)$ *Factorise to match $\ddot{x}$ with x equation

$\ddot{x} = -\frac{\pi^2}{9}\left(2\cos\left(\frac{\pi t}{3}\right) - 3\sin\left(\frac{\pi t}{3}\right)\right)$

$\ddot{x} = -\left(\frac{\pi}{3}\right)^2 x$ which is in the form of $a = -n^2x$

(Q2b) What is the initial displacement?

$x(0) = 2\cos\left(\frac{\pi \times 0}{3}\right) - 3\sin\left(\frac{\pi \times 0}{3}\right) =$ **2** metres.

(Q2c) What is the amplitude and period?

$A = \sqrt{2^2 + (-3)^2} \quad Period = 2\pi/b$

$A = \sqrt{13} =$ **3.61** m $\quad Period = 2\pi/(\pi/3) =$ **6** s

INCREMENTAL FORMULA

Small Change and Approximation

- Finds approximate change in y from a small change in x.

$$\delta y \approx \frac{dy}{dx} \times \delta x$$

Small Change by Euler's Method

Step 1	Determine dy/dx using implicit differentiation techniques.
Step 2	Select appropriate small change in x to use in dy/dx (*e.g.* $\delta x = 0.1$).
Step 3	Find value of δy by incrementally calculating dy/dx for each δx.

Incremental Formula Example

(Q1) $dy/dx = xy - x^2$ with a point at (5,6). Determine an estimate for y when $x = 5.2$.

- Using Euler's method with $\delta x = 0.1$

x	y	dy/dx	$\delta y \approx dy/dx \times \delta x$
5	6	5	0.5
5.1	6.5	7.14	0.714
5.2	7.214	N/A	N/A

∴ estimate for y when $x = 5.2$ is $y =$ **7.214**

RELATED RATES

Related Rates Notation

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dt}{dx}$	$\frac{dt}{dx} = 1 \div \frac{dx}{dt}$

Related Rates Examples

(Q1) Cylindrical balloon is inflated at a constant rate of $0.5\, m^3/min$ and has its height equal to its diameter. Find the rate of change of the surface area when it contains $2\, m^3$ of air.

- Finding expression of surface area:

$V = \pi r^2 h = \pi r^2(2r) = 2\pi r^3 \;\therefore r = \sqrt[3]{V/1\pi}$

$SA = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \times 2r = 6\pi r^2$

$SA = 6\pi \times \sqrt[3]{V^2/4\pi^2} = \sqrt[3]{54\pi V^2}$

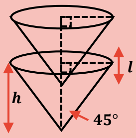
- Finding rate of change of surface area:

$\frac{dSA}{dV} = \frac{2}{3}\sqrt[3]{54\pi} \times V^{\frac{-1}{3}}$ and $\frac{dSA}{dt} = \frac{dSA}{dV} \times \frac{dV}{dt}$

$\frac{dSA}{dt} = \frac{2}{3}\sqrt[3]{54\pi} \times V^{\frac{-1}{3}} \times \frac{dV}{dt} = \frac{2}{3}\sqrt[3]{54\pi} \times 0.5^{\frac{-1}{3}} \times 2$

$\frac{dSA}{dt} =$ **5.86** m^2/min

(Q2) Shown on right is two identical circular cones each with height h cm and semi-vertical angle 45°. The lower cone is filled with water with the upper cone being lowered into it at a rate of $dl/dt = 8$ where time is in seconds. As upper cone is lowered, water spills out of the bottom cone that has V cm volume remaining.



(Q2a) Show that $V = \pi/3 \times (h^3 - l^3)$

The radius of the cones is h cm. The volume of water in the lower cone at time t is given by:

$V = \frac{\pi h^2 \times h}{3} - \frac{\pi l^2 \times l}{3} = \frac{\pi}{3}(h^3 - l^3)$ QED.

(Q2b) Find rate of change of V when the upper cone has been lowered by $3\, cm$ (*i.e.* $l = 3$).

$\frac{dV}{dt} = \frac{dV}{dl} \times \frac{dl}{dt} = -\pi l^2 \times 8 =$ **$-8\pi l^2$** cm^3/s

(Q2c) Find rate of change of V when the lower cone has lost 12.5% of its water in terms of h.

The lower cone has lost 12.5% of water when:

$\frac{\pi l^3}{3} = \frac{1}{8}\frac{\pi h^3}{3}$ which rearranging gives $h = 2l$

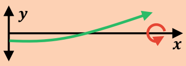
$\frac{dV}{dt} = \frac{dV}{dl} \times \frac{dl}{dt} = -8\pi l^2 =$ **$-2\pi h^2$** cm^3/s

VOLUMES OF REVOLUTION

Volumes of Revolution Formulae

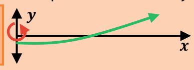
- Rotating a function 360° around the x or y-axis creates a three-dimensional solid.
- Volumes of revolution about the x-axis:
  - Upper and lower bounds on the x-axis.
  - y is the subject of equation in terms of x.

$$V = \pi \int_a^b y^2 dx$$



- Volumes of revolution about the y-axis:
  - Upper and lower bounds on the y-axis.
  - x is the subject of equation in terms of y.

$$V = \pi \int_a^b x^2 dy$$



Volumes of Revolution Examples

(Q1) Find the region bounded by the line $x = \frac{\pi}{2}$ and $y = 3\tan(x/3)$ rotated around the x-axis.

$V = \pi\int_a^b y^2 dx = \pi\int_0^{\pi/2}(3\tan(x/3))^2 dx$

$(3\tan(x/3))^2 = 9\tan^2(x/3) = 9\sec^2(x/3) + 9$

$= \pi\int_0^{\frac{\pi}{2}} 9\sec^2(x/3) - 9\, dx = \frac{-9\pi}{2} + 9\sqrt{3}\pi =$ **1.45**

(Q2) Determine the volume of the region in between the functions $x = y^2 - 6y + 10$ and $x = 5$ rotated around the y-axis.

- Determine the points of intersection:

$5 = y^2 - 6y + 10 \to 0 = y^2 - 6y + 5$

$0 = (y - 5)(y - 1) \to y = 1, 5$

Hence, points of intersection are (5,1) and (5,5)

Inner radius $= y^2 - 6y + 10$, outer radius $= 5$

∴ can treat this as an area between two curves question with respect to the y-axis.

$\therefore x^2 = [(outer\ radius)^2 - (inner\ radius)^2]$

$= [(5)^2 - (y^2 - 6y + 10)^2]$

$= [-75 + 120y - 56y^2 + 12y^3 - y^4]$

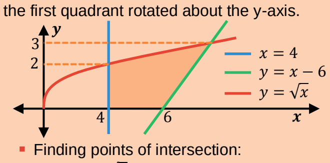
- Finding volume around y-axis:

$V = \pi\int_1^5 -75 + 120y - 56y^2 + 12y^3 - y^4 dy$

$= \pi\left[-75y + 60y^2 - \frac{56}{3}y^3 + 3y^4 - \frac{1}{5}y^5\right]_1^5$

$= 1088\pi/15 =$ **227.87**

(Q3) Write an expression for the volume of the solid generated by the area enclosed by $y = \sqrt{x}$, $y = 0$, $-x + y = -6$ and $x = 4$, lying in the first quadrant rotated about the y-axis.



- Finding points of intersection:

$x = 4$ and $y = \sqrt{x}$ intersect at $y = 4$

$-x + y = -6$ and $y = \sqrt{x}$ intersect at $y = 3$

- Finding volume around y-axis:

$\int_0^3 \pi(y + 6)^2 dy - \int_2^3 \pi(y^4) dy - \int_0^2 \pi(16) dy$

STATISTICAL INFERENCE

SAMPLES AND CONFIDENCE

Central Limit Theorem (*CLT*)

- If there are a large number of independent random samples (*i.e.* $n \geq 30$), the data can be modelled using a normal distribution.
- Also appropriate if np and $np(1 - p) \geq 10$.
- Uses sample size not number of samples.

CLT of a Random Variable X

- μ is population mean and $\bar{X}$ is sample mean.
- σ is population S.D. and s is sample S.D.
- If $n \geq 30$, $X \sim N$ with the following parameters:

Mean (stays)	S.D. (changes)	Z-Score (changes)
$\bar{X}$	$\frac{s}{\sqrt{n}}$	$Z = \frac{\bar{X} - \mu}{s/\sqrt{n}}$

Confidence Intervals (*CI*)

- Probability that confidence interval (at a certain level) will contain the population proportion.

$$(\bar{X} - sz/\sqrt{n},\ \bar{X} + sz/\sqrt{n}) = (CI_L, CI_U)$$

- Z : z-score for a given confidence interval.
- CI_L : confidence interval lower bound.
- CI_U : confidence interval upper bound.

- Commonly used Confidence Intervals:

% Confidence Interval	Z-Score
99% Confidence Interval	2.58
95% Confidence Interval	1.96
90% Confidence Interval	1.645

- ClassPad Main App Custom CI%:

$$z_{CI\%} = -1 \times invNormCDf("C", CI\%, 1, 0)$$

- Z : z-score for a given confidence interval.
- CI_L : confidence interval lower bound.
- CI_U : confidence interval upper bound.

- Find sample size for a confidence interval:

$$n = \left(\frac{z \times \sigma}{d}\right)^2$$

- d : value of the difference from the mean.

Statistical Inference Examples

(Q1) Find 95% confidence interval of a sample of 25 results with mean of 20 and variance of 4.

$20 - 1.96(2/\sqrt{25}) \leq \mu \leq 20 + 1.96(2/\sqrt{25})$

Hence, the 95% CI is **[19.216, 20.784]**

(Q2) What size sample is needed to ensure that sample mean is within 1.5 of population mean with 99% confidence, given the S.D. is 13.

$n = \left(\frac{z \times \sigma}{d}\right)^2 = \left(\frac{2.58 \times 13}{1.5}\right)^2 = 499.96 \approx$ **500**

(Q3) How large of a sample is needed to be 95% confident that the sample mean is within 10 of the population mean, given the S.D. is 15.

$10 = 1.96(15/\sqrt{n}) \to n = 8.6436 \approx$ **9**

(Q4) 45 samples of mean 94 and S.D. 12 was taken. Find parameters of the distribution:

Approximates to normal: $X \sim N\left(94, (12/\sqrt{45})^2\right)$

(Q5) S.D. of a population of water usage per month in households in a suburb is 1050 L. Random sample of 25 homes were made and total water used over a month was 260,000 L.

(Q5a) Find the parameters of the distribution:

Find mean usage: $\bar{X} = 260000/25 = 10400\, L$

$\therefore X \sim N\left(10400, (1050/\sqrt{25})^2\right)$

(Q5b) Find a 92% CI for mean water usage:

$z_{92\%} = -1 \times invNormCDf(C, 0.92, 1, 0) = 1.751$

$10400 \pm 1.751(1050/\sqrt{25}) =$ **[10032, 10767.7]**

(Q5c) If the water company repeated the random sampling process with 92% CI calculations from (Q5b) a total of 50 times, how many intervals would you expect to contain true pop. mean?

$92\%\ of\ 50 = 0.92 \times 50 =$ **46** of the CI intervals as the 92% refers to the 92% chance that the population mean is contained in the interval.

(Q6) The waiting time at the drive thru of a fast food restaurant is normally distributed with a mean waiting time of 5 mins and S.D. 3 mins.

(Q6a) A sample of 150 customers were taken and had a mean wait time of 6 mins. Is this sample significantly different at the 5% level?

5% level → 95% $CI \to z = 1.96$

$5 - 1.96(3/\sqrt{150}) \leq \bar{X} \leq 5 + 1.96(3/\sqrt{150})$

Hence, the 95% CI is **[4.4632, 5.5368]** which does not contain the sample mean, therefore there is a significant difference at the 5% level.

ATAR Math Specialist
Units 3 & 4 Exam Notes